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The Pseudothreshold Expansion of the 2-loop Sunrise Selfmass Master Amplitudes.

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Abstract

The values at pseudothreshold of two loop sunrise master amplitudes with arbitrary masses are obtained by solving a system of differential equations. The expansion at pseudothreshold of the amplitudes is constructed and some lowest terms are explicitly presented.

PACS 11.10.-z Field theory
PACS 11.10.Kk Field theories in dimensions other than four
PACS 11.15.Bt General properties of perturbation theory

1 Introduction.

The sunrise graph (also known as sunset or London transport diagram) appears naturally, as a consequence of tensorial reduction, in several higher order calculations in gauge theories. Due to the presence of heavy quarks, vector bosons and Higgs particles all the internal lines may carry a different mass, so that sunrise amplitudes depend in the general case on three internal masses $m_i,\ i = 1, 2, 3$, and the external scalar variable p^2, if p_μ is the external momentum (in n-dimensional Euclidean space).

For a proper understanding of their behaviour as well as for a check of the numerical calculations, it is convenient to know the amplitudes off-shell, and also around some particular values of p^2, such as $p^2 = 0$, $p^2 = \infty$, at the threshold and at the pseudothresholds.

This paper is devoted to the analytic evaluation of the coefficients of the expansion of the sunrise amplitudes in p^2, at the pseudothreshold value $p^2 = -(m_1 + m_2 - m_3)^2$ (the other pseudothreshold values $-(m_1 - m_2 + m_3)^2$, $-(m_1 - m_2 - m_3)^2$, can be easily obtained by permutation of the masses); the approach relies on the exploitation of the information contained in the linear system of first order differential equations in p^2, which is known to be satisfied by the sunrise amplitudes themselves [1]. It is to be noted that all the above points ($p^2 = 0,\ \infty$, threshold and pseudothresholds) correspond to the Fuchsian points of the differential equations, which therefore emerge as a natural tool for their discussion.

The analytic properties of Feynman diagrams at threshold and pseudothresholds are well known, see for example [2]. The sunrise diagram, with different masses, has been investigated in [3], while in [4] the values of the amplitudes at threshold and pseudothreshold were obtained. With the method established in [5] and [6], further, the expansion around threshold was recently obtained in [7].

The sunrise amplitudes are regular at any of the pseudothresholds, say s_0, so that they can be expanded as a single power series in $(p^2 - s_0)$ around that point [8]. When the expansions are inserted in the differential equations, the equations become a set of algebraic equations in the coefficients of the expansions; the obtained algebraic equations can then be solved recursively, for arbitrary value of the dimension n, once the initial conditions, *i.e.* the values of the sunrise amplitudes at the considered pseudothreshold, are given. As those initial values are in turn functions of the masses, we find in our approach a system of linear differential equations in the masses satisfied by them. We expand the equations in the masses in the dimension n around $n = 4$ and solve them explicitly up to the finite part in $(n - 4)$. This result for $n = 4$ and $p^2 = s_0$ is in

agreement with the literature [4].

We then look at the recursive solution of the algebraic equations for the coefficients of the expansion in $(p^2 - s_0)$, not yet discussed in the literature for the pseudothreshold (our results are given in the Sections 4 and 5). It turns out that the formula, expressing the first order coefficients of the $(p^2 - s_0)$ expansion in terms of the zeroth order values, involves the coefficient $1/(n-4)$. Therefore the finite part at $(n-4)$ of the first order terms in the $(p^2 - s_0)$ expansion involve the first order terms in $(n-4)$ of the zeroth order terms in $(p^2 - s_0)$. Rather than evaluating those first order terms in $(n-4)$, we prefer to evaluate directly the required finite part in n of the first order terms of the $(p^2 - s_0)$ expansion by means of a suitably subtracted dispersion relation in $(p^2 - s_0)$ at $n = 4$. Fortunately, the formulae expressing the higher order coefficients of the $(p^2 - s_0)$ expansion involve coefficients like $1/(n-5), 1/(n-6)$ etc., which are finite at $n = 4$, and can be used without further problems for evaluating those higher order terms.

The plan of the paper is as follows. In the second section the system of differential equations for the sunrise master amplitudes is recalled; in the third section the resulting differential equations in the masses for the values at the pseudothreshold are derived, expanded in $(n-4)$ and solved explicitly up to the finite part in $(n-4)$; in the fourth section the expansion in p^2 at the pseudothreshold is discussed; in the fifth section the finite part in $(n-4)$ of the first term in the p^2 expansion is given. The sixth section is a short summary, while the Appendix contains a list of various functions (mainly polynomials) used in the text.

2 The equations.

It is known that the two-loop sunrise self-mass graph with arbitrary masses m_1, m_2, m_3 has four independent master amplitudes [9], which are referred to, as in [1], by

$$F_\alpha(n, m_1^2, m_2^2, m_3^2, p^2), \qquad \alpha = 0, 1, 2, 3 \ , \tag{1}$$

where n is the continuous number of dimensions, $m_i, i = 1, 2, 3$ the three masses and p_μ the external n-momentum. $F_0(n, m_1^2, m_2^2, m_3^2, p^2)$ is the scalar amplitude

$$\begin{aligned} F_0(n, m_1^2, m_2^2, m_3^2, p^2) &= \\ \int \frac{d^n k_1}{(2\pi)^{n-2}} \int \frac{d^n k_2}{(2\pi)^{n-2}} &\frac{1}{(k_1^2 + m_1^2)(k_2^2 + m_2^2)((p - k_1 - k_2)^2 + m_3^2)} \ , \end{aligned} \tag{2}$$

while for $i = 1, 2, 3$

$$F_i(n, m_1^2, m_2^2, m_3^2, p^2) = -\frac{\partial}{\partial m_i^2} F_0(n, m_1^2, m_2^2, m_3^2, p^2) \ , \tag{3}$$

but the 4 amplitudes are otherwise independent (*i.e.* none of them can be expressed as a linear combinations of the others times rational functions of the arguments).

In [1] it was shown how to write for them a system of first order linear differential equations of the form

$$\begin{aligned} p^2 \frac{\partial}{\partial p^2} F_0(n, m_1^2, m_2^2, m_3^2, p^2) &= (n-3) F_0(n, m_1^2, m_2^2, m_3^2, p^2) + \sum_{i=1}^{3} m_i^2 F_i(n, m_1^2, m_2^2, m_3^2, p^2) \ , \\ p^2 \frac{\partial}{\partial p^2} F_i(n, m_1^2, m_2^2, m_3^2, p^2) &= \frac{1}{D(m_1^2, m_2^2, m_3^2, p^2)} \\ & \left[\sum_{\alpha=0}^{3} A_{i,\alpha}(n, m_1^2, m_2^2, m_3^2, p^2) F_\alpha(n, m_1^2, m_2^2, m_3^2, p^2) + B_i(n, m_1^2, m_2^2, m_3^2, p^2) \right] \ , \end{aligned} \tag{4}$$

where the $A_{i,\alpha}(n, m_1^2, m_2^2, m_3^2, p^2)$ are known polynomials of the arguments, and the functions $B_i(n, m_1^2, m_2^2, m_3^2, p^2)$, constituting the non homogeneous part of the equations, are also known, being the combinations of other known polynomials times the products of two one-denominator, one-loop vacuum amplitudes of two different masses, $T(n, m_i^2) T(n, m_j^2)$, where, as in [1],

$$T(n, m^2) = C(n) \frac{m^{n-2}}{(n-2)(n-4)} \ , \tag{5}$$

and $C(n)$ is a coefficient depending on n which at $n = 4$ takes the value $C(4) = 1$ (its explicit form, which is essentially irrelevant in practice, can be found in [1]).

The explicit form of the $A_{i,\alpha}(n, m_1^2, m_2^2, m_3^2, p^2), B_i(n, m_1^2, m_2^2, m_3^2, p^2)$, appearing in Eq.(4) is given in [1] and is not repeated here for the sake of brevity; for the following, it is sufficient to give the explicit expression of the polynomial $D(m_1^2, m_2^2, m_3^2, p^2)$, which reads

$$\begin{aligned} D(m_1^2, m_2^2, m_3^2, p^2) = \quad & \left[p^2 + (m_1 + m_2 + m_3)^2\right] \left[p^2 + (m_1 + m_2 - m_3)^2\right] \\ & \left[p^2 + (m_1 - m_2 + m_3)^2\right] \left[p^2 + (m_1 - m_2 - m_3)^2\right] \ . \end{aligned} \tag{6}$$

The above polynomial vanish when p^2 takes one of the three pseudothreshold values $-(m_1 + m_2 - m_3)^2$, $-(m_1 - m_2 + m_3)^2$, $-(m_1 - m_2 - m_3)^2$, and at the physical threshold $p^2 = -(m_1 + m_2 + m_3)^2$. It is known that the $F_i(n, m_1^2, m_2^2, m_3^2, p^2)$ and their p^2-derivatives

are regular at the pseudothresholds; therefore, the apparent pseudothreshold pole in the *r.h.s.* of Eq.(4) must be canceled by a corresponding zero in the numerator.

The recurrence relations solving the integration by part identities of [10] for the sunrise amplitudes can be also written (see [9] or the Appendix of [1]) in a form similar to Eq.(4), such as

$$\begin{aligned}\frac{\partial}{\partial m_3^2}F_i(n,m_1^2,m_2^2,m_3^2,p^2) &= \frac{1}{D(m_1^2,m_2^2,m_3^2,p^2)}\\ &\left[\sum_{\alpha=0}^{3}R_{i,\alpha}(n,m_1^2,m_2^2,m_3^2,p^2)F_\alpha(n,m_1^2,m_2^2,m_3^2,p^2)+S_i(n,m_1^2,m_2^2,m_3^2,p^2)\right] , \end{aligned} \tag{7}$$

where $i=1,2,3$, the $R_{i,\alpha}(n,m_1^2,m_2^2,m_3^2,p^2)$ and the $S_i(n,m_1^2,m_2^2,m_3^2,p^2)$ have the same structure as the $A_{i,\alpha}(n,m_1^2,m_2^2,m_3^2,p^2)$ and the $B_i(n,m_1^2,m_2^2,m_3^2,p^2)$ of Eq.(4); as in Eq.(4), the apparent pseudothreshold singularity of Eq.(7) must be canceled by a corresponding zero of the numerator.

3 The pseudothreshold values of the master amplitudes.

As the master amplitudes are regular at the pseudothresholds, considering for definiteness the pseudothreshold at $p^2=-(m_1+m_2-m_3)^2$, we can expand $F_0(n,m_1^2,m_2^2,m_3^2,p^2)$ in p^2 around that point as

$$\begin{aligned}F_\alpha(n,m_1^2,m_2^2,m_3^2,p^2) &= H^{(\alpha,0)}(n,m_1,m_2,m_3)\\ &+ H^{(\alpha,1)}(n,m_1,m_2,m_3)\left(p^2+(m_1+m_2-m_3)^2\right)\\ &+ H^{(\alpha,2)}(n,m_1,m_2,m_3)\left(p^2+(m_1+m_2-m_3)^2\right)^2+...,\\ & \quad \alpha=0,1,2,3 ,\end{aligned} \tag{8}$$

from which we can write at once, by using the definitions Eq.(3),

$$\begin{aligned}G_0(n,m_1,m_2,m_3) &\equiv F_0\left(n,m_1^2,m_2^2,m_3^2,p^2=-(m_1+m_2-m_3)^2\right)\\ &= H^{(0,0)}(n,m_1,m_2,m_3) ,\\ G_1(n,m_1,m_2,m_3) &\equiv F_1\left(n,m_1^2,m_2^2,m_3^2,p^2=-(m_1+m_2-m_3)^2\right)\\ &= H^{(1,0)}(n,m_1,m_2,m_3)\\ &= -\frac{\partial}{\partial m_1^2}H^{(0,0)}(n,m_1,m_2,m_3)-\frac{m_1+m_2-m_3}{m_1}H^{(0,1)}(n,m_1,m_2,m_3) ,\end{aligned}$$

$$
\begin{aligned}
G_2(n,m_1,m_2,m_3) &\equiv F_2\left(n,m_1^2,m_2^2,m_3^2,p^2=-(m_1+m_2-m_3)^2\right)\\
&= H^{(2,0)}(n,m_1,m_2,m_3)\\
&= -\frac{\partial}{\partial m_2^2}H^{(0,0)}(n,m_1,m_2,m_3)-\frac{m_1+m_2-m_3}{m_2}H^{(0,1)}(n,m_1,m_2,m_3)\ ,\\
G_3(n,m_1,m_2,m_3) &\equiv F_3\left(n,m_1^2,m_2^2,m_3^2,p^2=-(m_1+m_2-m_3)^2\right)\\
&= H^{(3,0)}(n,m_1,m_2,m_3)\\
&= -\frac{\partial}{\partial m_3^2}H^{(0,0)}(n,m_1,m_2,m_3)+\frac{m_1+m_2-m_3}{m_3}H^{(0,1)}(n,m_1,m_2,m_3)\ .
\end{aligned}
\tag{9}
$$

By inserting the above expansions in the four equations Eq.(4) and in the three equations Eq.(7) for $i=1,2,3$, we find only five algebraically independent equations for $G_\alpha(n,m_1,m_2,m_3), \alpha=0,1,2,3$ and their derivatives in respect to m_3, which can be written as

$$
\begin{aligned}
&-\frac{(n-3)(3n-8)}{2}G_0(n,m_1,m_2,m_3)=\\
&+(n-3)m_1(2m_1-m_3+m_2)G_1(n,m_1,m_2,m_3)+\frac{(n-2)^2}{4m_2m_3}T(n,m_2^2)T(n,m_3^2)\\
&+(n-3)m_2(2m_2-m_3+m_1)G_2(n,m_1,m_2,m_3)+\frac{(n-2)^2}{4m_1m_3}T(n,m_1^2)T(n,m_3^2)\\
&+(n-3)m_3(2m_3-m_1-m_2)G_3(n,m_1,m_2,m_3)-\frac{(n-2)^2}{4m_1m_2}T(n,m_1^2)T(n,m_2^2)\ ,
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&\frac{m_3-m_1-m_2}{2}\frac{\partial}{\partial m_3}G_0(n,m_1,m_2,m_3)=(n-3)G_0(n,m_1,m_2,m_3)\\
&+m_1^2G_1(n,m_1,m_2,m_3)+m_2^2G_2(n,m_1,m_2,m_3)+m_3(m_1+m_2)G_3(n,m_1,m_2,m_3)
\end{aligned}
\tag{11}
$$

and, for $i=1,2,3$,

$$
\begin{aligned}
&(m_3-m_1-m_2)(m_3-m_1)(m_3-m_2)\frac{\partial}{\partial m_3}G_i(n,m_1,m_2,m_3)=\\
&P_{i,1}(n,m_1,m_2,m_3)G_1(n,m_1,m_2,m_3)+P_{i,2}(n,m_1,m_2,m_3)G_2(n,m_1,m_2,m_3)\\
&+P_{i,3}(n,m_1,m_2,m_3)G_3(n,m_1,m_2,m_3)+Q_{i,1}(n,m_1,m_2,m_3)T(n,m_2^2)T(n,m_3^2)\\
&+Q_{i,2}(n,m_1,m_2,m_3)T(n,m_1^2)T(n,m_3^2)+Q_{i,3}(n,m_1,m_2,m_3)T(n,m_1^2)T(n,m_2^2)\ ,
\end{aligned}
\tag{12}
$$

where $T(n, m^2)$ is defined in Eq.(5), while the rational functions $P_{i,j}(n, m_1, m_2, m_3)$ and $Q_{i,j}(n, m_1, m_2, m_3)$ (which differ from the functions of the same name, but different arguments, appearing in [1]) are defined as

$$
\begin{aligned}
P_{1,1}(n, m_1, m_2, m_3) &= -\frac{n-3}{2} m_2(m_3 - m_2) - (m_3 - m_1)(m_3 - m_2) \\
P_{1,2}(n, m_1, m_2, m_3) &= \frac{n-3}{2} m_2(m_3 - m_1) \\
P_{1,3}(n, m_1, m_2, m_3) &= -(n-3) \left(m_3(m_3 - m_1 - m_2) + \frac{1}{2} m_2(m_1 + m_2) \right) \\
P_{2,1}(n, m_1, m_2, m_3) &= \frac{n-3}{2} m_1(m_3 - m_2) \\
P_{2,2}(n, m_1, m_2, m_3) &= -\frac{n-3}{2} m_1(m_3 - m_1) - (m_3 - m_1)(m_3 - m_2) \\
P_{2,3}(n, m_1, m_2, m_3) &= -(n-3) \left(m_3(m_3 - m_1 - m_2) + \frac{1}{2} m_1(m_1 + m_2) \right) \\
P_{3,1}(n, m_1, m_2, m_3) &= -(n-3) \frac{m_1}{m_3} (m_3 - m_2) \left(m_3 - \frac{1}{2} m_1 - m_2 \right) \\
P_{3,2}(n, m_1, m_2, m_3) &= -(n-3) \frac{m_2}{m_3} (m_3 - m_1) \left(m_3 - m_1 - \frac{1}{2} m_2 \right) \\
P_{3,3}(n, m_1, m_2, m_3) &= \frac{1}{m_3} (m_3 - m_1)(m_3 - m_2)(m_1 + m_2 - 2m_3) \\
&+ \frac{n-3}{2m_3} \left(4(m_3 - m_1)(m_3 - m_2)(m_3 - m_1 - m_2) + m_1 m_2 (m_1 + m_2 - 2m_3) \right) \ , \quad (13)
\end{aligned}
$$

and

$$
\begin{aligned}
Q_{1,1}(n, m_1, m_2, m_3) &= -\frac{(n-2)^2}{8 m_2 m_3^2} (m_3 - m_2) \\
Q_{1,2}(n, m_1, m_2, m_3) &= \frac{(n-2)^2}{4 m_1^2 m_3^2} (m_3 - m_1)(m_3 - \frac{1}{2} m_2) \\
Q_{1,3}(n, m_1, m_2, m_3) &= -\frac{(n-2)^2}{8 m_1^2 m_2} (m_2 - m_1) \\
Q_{2,1}(n, m_1, m_2, m_3) &= \frac{(n-2)^2}{4 m_2^2 m_3^2} (m_3 - m_2)(m_3 - \frac{1}{2} m_1) \\
Q_{2,2}(n, m_1, m_2, m_3) &= -\frac{(n-2)^2}{8 m_1 m_3^2} (m_3 - m_1) \\
Q_{2,3}(n, m_1, m_2, m_3) &= -\frac{(n-2)^2}{8 m_1 m_2^2} (m_1 - m_2)
\end{aligned}
$$

$$
\begin{aligned}
Q_{3,1}(n,m_1,m_2,m_3) &= \frac{(n-2)^2}{4m_2m_3^3}(m_3-m_2)(m_3-\frac{1}{2}m_1) \\
Q_{3,2}(n,m_1,m_2,m_3) &= \frac{(n-2)^2}{4m_1m_3^3}(m_3-m_1)(m_3-\frac{1}{2}m_2) \\
Q_{3,3}(n,m_1,m_2,m_3) &= -\frac{(n-2)^2}{4m_1m_2m_3}(m_3-\frac{1}{2}m_1-\frac{1}{2}m_2)\ .
\end{aligned} \tag{14}
$$

The three Eq.(12) form a system of equations involving only the master amplitudes $G_i(n,m_1,m_2,m_3)\quad i=1,2,3$ and not $G_0(n,m_1,m_2,m_3)$, while Eq.(10) can be regarded as an identity expressing $G_0(n,m_1,m_2,m_3)$ in terms of the $G_i(n,m_1,m_2,m_3)$; that means that only three of the four master amplitudes are independent at the pseudothreshold and, as a consequence, Eq.(11) is implied by the other four equations and is not an independent equation.

As any system of k linear first order equations is equivalent to a single k-th order equation, we transform the system Eq.(12) into a single third order differential equation for a single unknown function, which we choose to be $G_0(n,m_1,m_2,m_3)$; further, we use Eq.(10) and Eq.(11) for expressing the $G_i(n,m_1,m_2,m_3)$ in terms of $G_0(n,m_1,m_2,m_3)$ and its derivatives with respect to m_3. Once the equation for $G_0(n,m_1,m_2,m_3)$, is solved, *i.e.* $G_0(n,m_1,m_2,m_3)$ is known explicitly, the $G_i(n,m_1,m_2,m_3)$ can be easily recovered, as we show later.

Before continuing, we expand all the functions in n around $n=4$; according to the notation of [1] we write

$$
\begin{aligned}
G_\alpha(n,m_1,m_2,m_3) \quad &= \quad C^2(n)\Big[\frac{1}{(n-4)^2}G_\alpha^{(-2)}(m_1,m_2,m_3)+\frac{1}{(n-4)}G_\alpha^{(-1)}(m_1,m_2,m_3) \\
&\quad +G_\alpha^{(0)}(m_1,m_2,m_3)+O(n-4)\Big] \quad \alpha=0,1,2,3\ .
\end{aligned} \tag{15}
$$

The function $C(n)$ has been already introduced above; again from [1], where the singular parts of $F_\alpha(n,m_1^2,m_2^2,m_3^2,p^2)$ are given for arbitrary values of p^2, we have

$$
\begin{aligned}
G_0^{(-2)}(m_1,m_2,m_3) &= -\frac{1}{8}(m_1^2+m_2^2+m_3^2)\ , \\
G_0^{(-1)}(m_1,m_2,m_3) &= -\frac{1}{32}(m_1+m_2-m_3)^2+\frac{3}{16}(m_1^2+m_2^2+m_3^2) \\
&\quad -\frac{1}{8}\left[m_1^2\log(m_1^2)+m_2^2\log(m_2^2)+m_3^2\log(m_3^2)\right]\ , \\
G_i^{(-2)}(m_1,m_2,m_3) &= \frac{1}{8}\ \ ,\ \ G_i^{(-1)}(m_1,m_2,m_3) = -\frac{1}{16}+\frac{1}{8}\log(m_i^2)\ \ ,\ \ i=1,2,3.
\end{aligned} \tag{16}
$$

The third order differential equation for $G_0^{(0)}(m_1,m_2,m_3)$ then reads

$$
\begin{aligned}
& g_3(m_1,m_2,m_3)\frac{\partial^3}{\partial m_3^3}G_0^{(0)}(m_1,m_2,m_3) + g_2(m_1,m_2,m_3)\frac{\partial^2}{\partial m_3^2}G_0^{(0)}(m_1,m_2,m_3) \\
& + g_1(m_1,m_2,m_3)\frac{\partial}{\partial m_3}G_0^{(0)}(m_1,m_2,m_3) + g_0(m_1,m_2,m_3)G_0^{(0)}(m_1,m_2,m_3) \\
& + g(m_1,m_2,m_3) = 0 \ , \qquad (17)
\end{aligned}
$$

were the $g_\alpha(m_1,m_2,m_3) \quad \alpha=0,..,3$ are the following polynomials

$$
\begin{aligned}
g_3(m_1,m_2,m_3) =& m_3(m_3-m_1)(m_3-m_2)x^3 \\
g_2(m_1,m_2,m_3) =& (m_3-m_1)(m_3-m_2)\left[5m_3+\frac{1}{2}(m_1+m_2)\right]x^2 \\
g_1(m_1,m_2,m_3) =& -2x^4+2(m_1+m_2)x^3+(3m_1^2+8m_1m_2+3m_2^2)x^2+4m_1m_2(m_1+m_2)x \\
g_0(m_1,m_2,m_3) =& -6\left(m_3-\frac{1}{2}(m_1+m_2)\right)\left(m_3^2-m_3(m_1+m_2)+\frac{1}{3}m_1m_2\right) \ , \qquad (18)
\end{aligned}
$$

with $x=m_3-m_1-m_2$, while the function $g(m_1,m_2,m_3)$, which is written in the Appendix, is a combination of similar polynomials and of first and second powers of the logarithms of the masses generated in the expansion in $(n-4)$ of the $T(n,m^2)$. In order to solve Eq.(17), we make the ansatz

$$
G_0^{(0)}(m_1,m_2,m_3) = a(m_1,m_2,m_3)G_0^{(0,1)}(m_1,m_2,m_3) \qquad (19)
$$

with the idea of fixing the otherwise undetermined function $a(m_1,m_2,m_3)$ by imposing that in the resulting differential equation for $G_0^{(0,1)}(m_1,m_2,m_3)$ the coefficient of the term without derivatives vanishes, *i.e.* by imposing

$$
\begin{aligned}
& g_3(m_1,m_2,m_3)\frac{\partial^3}{\partial m_3^3}a(m_1,m_2,m_3) + g_2(m_1,m_2,m_3)\frac{\partial^2}{\partial m_3^2}a(m_1,m_2,m_3) \\
& + g_1(m_1,m_2,m_3)\frac{\partial}{\partial m_3}a(m_1,m_2,m_3) + g_0(m_1,m_2,m_3)a(m_1,m_2,m_3) = 0 \ , \qquad (20)
\end{aligned}
$$

which of course is nothing but the homogeneous version of Eq.(17). We further look for $a(m_1,m_2,m_3)$ in the form of a rational function in m_3 and m_1,m_2; by trial and error we find that Eq.(19) can be given the explicit form

$$
G_0^{(0)}(m_1,m_2,m_3) = -\frac{2m_3-m_1-m_2}{2(m_3-m_1-m_2)^2}G_0^{(0,1)}(m_1,m_2,m_3). \qquad (21)
$$

With the above substitution Eq.(17) has taken the form of a second order differential equation in the first m_3-derivative (the essential property of the transformation Eq.(21)

is that $G_0^{(0,1)}(m_1,m_2,m_3)$ does not appear anymore in the equation). By iteration of the method, the following chain of substitutions is obtained

$$\begin{aligned}
\frac{\partial}{\partial m_3}G_0^{(0,1)}(m_1,m_2,m_3) &= G_0^{(0,2)}(m_1,m_2,m_3) \\
G_0^{(0,2)}(m_1,m_2,m_3) &= \left[\frac{1}{4}(2m_3-m_1-m_2)^2+a_0+\frac{a_1}{(2m_3-m_1-m_2)^2}\right] \\
&\quad G_0^{(0,3)}(m_1,m_2,m_3) \\
\frac{\partial}{\partial m_3}G_0^{(0,3)}(m_1,m_2,m_3) &= G_0^{(0,4)}(m_1,m_2,m_3) \\
G_0^{(0,4)}(m_1,m_2,m_3) &= \frac{1}{2}a_2 G_0^{(0,5)}(m_1,m_2,m_3)
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
a_0 &= -\frac{1}{6}(3m_1^2+2m_1m_2+3m_2^2) \\
a_1 &= \frac{1}{12}(m_1+3m_2)(3m_1+m_2)(m_1-m_2)^2 \\
a_2 &= \frac{(2m_3-m_1-m_2)\sqrt{m_3(m_3-m_1-m_2)}}{(m_3-m_1)^2(m_3-m_2)^2[m_3^2-(m_1+m_2)m_3-\frac{1}{3}m_1m_2]^2}\ ;
\end{aligned} \tag{23}$$

$G_0^{(0,5)}(m_1,m_2,m_3)$, finally, satisfies the following first order differential equation

$$\begin{aligned}
\frac{\partial}{\partial m_3}G_0^{(0,5)}(m_1,m_2,m_3) &= (m_3(m_3-m_1-m_2))^{-\frac{3}{2}}\Big[q_0+q_1\log(m_3)+q_2\log^2(m_3)\Big] \\
&\quad +(m_3(m_3-m_1-m_2))^{-\frac{1}{2}}\Big[q_{-1,0}m_3^{-1}+q_{-1,1}m_3^{-1}\log(m_3) \\
&\quad +q_{-1,2}m_3^{-1}\log^2(m_3)+q_{0,0}+q_{0,1}\log(m_3)+q_{0,2}\log^2(m_3) \\
&\quad +q_{1,0}m_3+q_{1,1}m_3\log(m_3)+q_{1,2}m_3\log^2(m_3) \\
&\quad +q_{2,0}m_3^2+q_{2,1}m_3^2\log(m_3)+q_{2,2}m_3^2\log^2(m_3) \\
&\quad +q_{3,0}m_3^3+q_{3,1}m_3^3\log(m_3)+q_{3,2}m_3^3\log^2(m_3) \\
&\quad +q_{4,0}m_3^4+q_{4,1}m_3^4\log(m_3)+q_{5,0}m_3^5+q_{5,1}m_3^5\log(m_3)\Big]
\end{aligned} \tag{24}$$

where the coefficients $q_\alpha,\ q_{\alpha,\beta}$ are functions of m_1 and m_2 only and are reported in the Appendix.

Eq.(24) is lengthy, but in fact it is a very simple differential equation, which can immediately be solved by quadrature. From Eq.(22) and Eq.(24) is obvious that we need to integrate three times in m_3 to get $G_0^{(0)}(m_1,m_2,m_3)$. The integrations are elementary

and the integration constants can be fixed at $m_3 = m_1 + m_2$; indeed, we are considering the pseudothreshold value $p^2 = -(m_1+m_2-m_3)^2$, which for that value of m_3 corresponds to $p^2 = 0$, where the master integrals of the sunrise graph are known in closed analytic form [1]. Once the integration constants are fixed, the explicit form of $G_0^{(0)}(m_1, m_2, m_3)$ reads

$$
\begin{aligned}
&G_0^{(0)}(m_1, m_2, m_3) = \\
&-\frac{1}{8(m_3 - m_1 - m_2)^2}\Big[m_1^3(m_1 + 2m_2)\mathcal{L}(m_1, m_2, m_3) + m_2^3(2m_1 + m_2)\mathcal{L}(m_2, m_1, m_3)\Big] \\
&-\frac{1}{4(m_3 - m_1 - m_2)}\Big[(m_1 + m_2)\Big(m_1^2\mathcal{L}(m_1, m_2, m_3) + m_2^2\mathcal{L}(m_2, m_1, m_3)\Big) \\
&+\frac{1}{2}(m_1^2 + m_2^2 + m_1 m_2)\left(m_1 \log\left(\frac{m_3}{m_1}\right) + m_2 \log\left(\frac{m_3}{m_2}\right)\right)\Big] \\
&+\frac{1}{8}(m_3^2 - m_1^2 - m_2^2)\left(\mathcal{L}(m_1, m_2, m_3) + \mathcal{L}(m_2, m_1, m_3)\right) \\
&-\frac{1}{32}\Big[m_1^2 \log^2(m_1^2) + m_2^2 \log^2(m_2^2) + m_3^2 \log^2(m_3^2) + (m_1^2 + m_2^2 - m_3^2)\log(m_1^2)\log(m_2^2) \\
&+(m_1^2 - m_2^2 + m_3^2)\log(m_1^2)\log(m_3^2) + (-m_1^2 + m_2^2 + m_3^2)\log(m_2^2)\log(m_3^2) \\
&-m_1(7m_1 + 2m_3)\log(m_1^2) - m_2(7m_2 + 2m_3)\log(m_2^2) \\
&+(2m_1^2 + 2m_1 m_2 + 2m_2^2 - 5m_3^2)\log(m_3^2)\Big] \\
&-\frac{11}{128}(m_1^2 + m_2^2 + m_3^2) - \frac{13}{64}(-m_1 m_2 + m_1 m_3 + m_2 m_3)
\end{aligned}
\tag{25}
$$

where

$$
\mathcal{L}(m_1, m_2, m_3) = \mathrm{Li}_2\left(1 - \frac{m_3}{m_2}\right) - \mathrm{Li}_2\left(-\frac{m_1}{m_2}\right) + \log\left(\frac{m_3}{m_1 + m_2}\right)\log\left(\frac{m_1}{m_2}\right) \tag{26}
$$

The above result is in agreement with [4]. To check it we have simply to expand in n around $n = 4$ our overall coefficient $C(n)$ (Eq.(9) of [1]) and the n-dependent overall coefficients of [4]).

As already mentioned, once $G_0(n, m_1, m_2, m_3)$ is known in closed analytic form, its derivatives are also known, and the other amplitudes $G_1(n, m_1, m_2, m_3)$, $G_2(n, m_1, m_2, m_3)$ and $G_3(n, m_1, m_2, m_3)$ can be easily obtained through simple relations with them.

To obtain such relations, we differentiate $G_0(n, m_1, m_2, m_3)$ in its integral representation with respect to m_1 and m_2, as done already in Eq.(11) for m_3, and we get after some algebra the following two additional relations

$$
\begin{aligned}
&\frac{\partial}{\partial m_1} G_0(n,m_1,m_2,m_3) = \frac{-2}{m_3-m_1-m_2}\Big[(n-3)G_0(n,m_1,m_2,m_3) \\
&+ m_1(m_3-m_2)G_1(n,m_1,m_2,m_3) + m_2^2 G_2(n,m_1,m_2,m_3) + m_3^2 G_3(n,m_1,m_2,m_3)\Big] \\
&\frac{\partial}{\partial m_2} G_0(n,m_1,m_2,m_3) = \frac{-2}{m_3-m_1-m_2}\Big[(n-3)G_0(n,m_1,m_2,m_3) \\
&+ m_1^2 G_1(n,m_1,m_2,m_3) + m_2(m_3-m_1)G_2(n,m_1,m_2,m_3) + m_3^2 G_3(n,m_1,m_2,m_3)\Big]
\end{aligned} \tag{27}
$$

The first of the Eq.(12), Eq.(10) and Eq.(27) form a system of equations, which allows to express master amplitudes $G_\alpha(n,m_1,m_2,m_3), \alpha = 0,1,2,3$, as a function of first derivatives of $G_0(n,m_1,m_2,m_3)$ only. As $G_0(n,m_1,m_2,m_3)$ is already known, we can express $G_i(n,m_1,m_2,m_3), i = 1,2,3$, as a function of $G_0(n,m_1,m_2,m_3)$ and two of its first derivatives (we choose derivatives in respect to m_1 and m_2). The solution reads (the expression for $G_2(n,m_1,m_2,m_3)$ can be easily found from $G_1(n,m_1,m_2,m_3)$ by the substitution $m_1 \to m_2$, $m_2 \to m_1$)

$$
\begin{aligned}
&G_1(n,m_1,m_2,m_3) = \\
&\frac{1}{8m_1m_3}\Bigg\{(m_1-3m_3)\frac{\partial}{\partial m_1}G_0(n,m_1,m_2,m_3) + (m_2+m_3)\frac{\partial}{\partial m_2}G_0(n,m_1,m_2,m_3) \\
&+ \frac{1}{m_3-m_1-m_2}\Bigg[\Big(2(n-3)(m_1+m_2)-(n-4)m_3\Big)G_0(n,m_1,m_2,m_3) \\
&+ \frac{(n-2)^2}{2(n-3)}\left(\frac{1}{m_2}T(n,m_2^2)T(n,m_3^2) + \frac{1}{m_1}T(n,m_1^2)T(n,m_3^2) - \frac{m_3}{m_1m_2}T(n,m_1^2)T(n,m_2^2)\right)\Bigg]\Bigg\} \\
&G_3(n,m_1,m_2,m_3) = \\
&\frac{1}{8m_3^2}\Bigg\{(3m_1-m_3)\frac{\partial}{\partial m_1}G_0(n,m_1,m_2,m_3) + (3m_2-m_3)\frac{\partial}{\partial m_2}G_0(n,m_1,m_2,m_3) \\
&+ \frac{1}{m_3-m_1-m_2}\Bigg[\Big((n-3)(6m_1+6m_2-7m_3)-m_3\Big)G_0(n,m_1,m_2,m_3) \\
&- \frac{(n-2)^2}{2(n-3)}\left(\frac{1}{m_2}T(n,m_2^2)T(n,m_3^2) + \frac{1}{m_1}T(n,m_1^2)T(n,m_3^2) - \frac{m_3}{m_1m_2}T(n,m_1^2)T(n,m_2^2)\right)\Bigg]\Bigg\}
\end{aligned} \tag{28}
$$

Expanding around $n = 4$ we find that $G_\alpha^{(-2)}(m_1,m_2,m_3)$ and $G_\alpha^{(-1)}(m_1,m_2,m_3)$, ($\alpha = 0,1,2,3$), already explicitly given in Eq.(16), automatically satisfy the Eq.(28), while for

$G_i^{(0)}(m_1,m_2,m_3)$, $(i=1,2,3)$, we find the explicit expressions

$$
\begin{aligned}
& G_1^{(0)}(m_1,m_2,m_3) = \\
& \quad \frac{1}{8(m_3-m_1-m_2)^2}\Big[m_1(m_1+2m_2)\mathcal{L}(m_1,m_2,m_3) - m_2^2\mathcal{L}(m_2,m_1,m_3)\Big] \\
& \quad +\frac{1}{4(m_3-m_1-m_2)}\left[(m_1+m_2)\mathcal{L}(m_1,m_2,m_3) + \frac{1}{2}\left(m_1\log\left(\frac{m_3}{m_1}\right) + m_2\log\left(\frac{m_3}{m_2}\right)\right)\right] \\
& \quad +\frac{1}{8}\Big(\mathcal{L}(m_1,m_2,m_3)+\mathcal{L}(m_2,m_1,m_3)\Big) + \frac{1}{32}\Big[\log^2(m_1^2) + \log(m_1^2)\log(m_2^2) \\
& \quad +\log(m_1^2)\log(m_3^2) - \log(m_2^2)\log(m_3^2) - 4\log(m_1^2) + 2\log(m_3^2) - 1\Big] \\
& G_3^{(0)}(m_1,m_2,m_3) = \\
& \quad \frac{1}{8(m_3-m_1-m_2)^2}\Big[m_1^2\mathcal{L}(m_1,m_2,m_3) + m_2^2\mathcal{L}(m_2,m_1,m_3)\Big] \\
& \quad +\frac{1}{8(m_3-m_1-m_2)}\left(m_1\log\left(\frac{m_3}{m_1}\right) + m_2\log\left(\frac{m_3}{m_2}\right)\right) \\
& \quad -\frac{1}{8}\Big(\mathcal{L}(m_1,m_2,m_3)+\mathcal{L}(m_2,m_1,m_3)\Big) + \frac{1}{32}\Big[\log^2(m_3^2) - \log(m_1^2)\log(m_2^2) \\
& \quad +\log(m_1^2)\log(m_3^2) + \log(m_2^2)\log(m_3^2) - 2\log(m_3^2) - 1\Big] \ ,
\end{aligned} \tag{29}
$$

where $\mathcal{L}(m_1,m_2,m_3)$ is defined by Eq.(26). The function $G_2^{(0)}(m_1,m_2,m_3)$ is obtained by substituting $m_1 \to m_2$, $m_2 \to m_1$ in the expression of $G_1^{(0)}(m_1,m_2,m_3)$.

4 The expansion at the pseudothreshold.

As recalled in [1], it is easy to find the coefficients of the expansion of a function in any variable once a system of differential equations in that variable is given. In this section we obtain the expansions in p^2 at the pseudothreshold of all the master integrals Eq.(1), Eq.(8). Putting expansions Eq.(8) into the system of equations Eq.(4) we get (comparing the lowest terms in expansion) from the first equation

$$
\begin{aligned}
H^{(0,1)}(n,m_1,m_2,m_3) \;\; &= \;\; -\frac{1}{(m_3-m_1-m_2)^2}\Big[(n-3)\,G_0(n,m_1,m_2,m_3) \\
& + m_1^2 G_1(n,m_1,m_2,m_3) + m_2^2 G_2(n,m_1,m_2,m_3) + m_3^2 G_3(n,m_1,m_2,m_3)\Big] \ .
\end{aligned} \tag{30}
$$

Substituting for the $G_\alpha(n,m_1,m_2,m_3)$ their expressions expanded around $n=4$, we have

$$
H^{(0,1)}(n,m_1,m_2,m_3) = C^2(n)\left[\frac{1}{32\ (n-4)} + H^{(0,1)}(n=4,m_1,m_2,m_3) + \mathcal{O}(n-4)\right] \ , \tag{31}
$$

with

$$
\begin{aligned}
H^{(0,1)}(n=4,m_1,m_2,m_3) \quad &= \\
&- \frac{1}{8(m_3-m_1-m_2)^4}\left[m_1^3(m_1+2m_2)\mathcal{L}(m_1,m_2,m_3)+m_2^3(2m_1+m_2)\mathcal{L}(m_2,m_1,m_3)\right] \\
&+ \frac{1}{8(m_3-m_1-m_2)^3}\Bigg[-2(m_1+m_2)\left(m_1^2\mathcal{L}(m_1,m_2,m_3)+m_2^2\mathcal{L}(m_2,m_1,m_3)\right) \\
&\qquad + m_1m_2(m_1+m_2)\log\left(\frac{m_1m_2}{m_3^2}\right)+m_1^3\log\left(\frac{m_1}{m_3}\right)+m_2^3\log\left(\frac{m_2}{m_3}\right)\Bigg] \\
&- \frac{1}{8(m_3-m_1-m_2)^2}\Bigg[m_1^2\mathcal{L}(m_1,m_2,m_3)+m_2^2\mathcal{L}(m_2,m_1,m_3)-\frac{1}{2}\left(m_1^2+m_1m_2+m_2^2\right) \\
&\qquad - m_1\left(\frac{3}{2}m_1+m_2\right)\log\left(\frac{m_1}{m_3}\right)-m_2\left(m_1+\frac{3}{2}m_2\right)\log\left(\frac{m_2}{m_3}\right)\Bigg] \\
&+ \frac{m_1+m_2}{16(m_3-m_1-m_2)} \quad + \frac{1}{16}\log(m_3)-\frac{5}{128}\ . \qquad (32)
\end{aligned}
$$

The other three equations form a system of linear equations, which can be easily solved algebraically. The solution for $H^{(1,1)}(n,m_1,m_2,m_3)$ reads

$$
\begin{aligned}
H^{(1,1)}(n,m_1,m_2,m_3) \quad &= \quad -\frac{1}{m_1(m_3-m_1-m_2)^2}\Bigg[\\
&\frac{1}{n-4}\Bigg(\frac{1}{16}\left(m_3-2m_1-m_2\right)G_1(n,m_1,m_2,m_3) \\
&\quad + \frac{1}{16}\left(m_3-m_1-2m_2\right)G_2(n,m_1,m_2,m_3) \\
&\quad + \frac{1}{16}\left(2m_3-m_1-m_2\right)G_3(n,m_1,m_2,m_3)\Bigg)+\cdots\Bigg]\ . \qquad (33)
\end{aligned}
$$

We do not put here the rest of the expression as it is not relevant for the subsequent discussion. Similar formulae are found for $H^{(2,1)}(n,m_1,m_2,m_3)$ and $H^{(3,1)}(n,m_1,m_2,m_3)$.

As it is evident from the Eq.(33), to get terms $\sim (n-4)^0$ in the $(n-4)$ expansion of $H^{(1,1)}(n,m_1,m_2,m_3)$ we need terms $\sim (n-4)$ of the $G_i(n,m_1,m_2,m_3)$, $i=1,2,3$, (a similar problem was found in [7] for the expansion at the threshold). The terms $\sim (n-4)$ are not yet known and could in principle be investigated with the method used in the previous section. The integrals obtained that way contain up to trilogarithmic functions (which are however expected to cancel out in the combination appearing in the *r.h.s.* of Eq.(33) - see below), so we decided to get $H^{(i,1)}(n,m_1,m_2,m_3)$, $i=1,2,3$, in a simpler

way. By using Eq.(3) we get the following relations

$$
\begin{aligned}
H^{(1,1)}(n,m_1,m_2,m_3) &= -\frac{\partial H^{(0,1)}(n,m_1,m_2,m_3)}{\partial m_1^2} + 2\frac{m_3-m_1-m_2}{m_1} H^{(0,2)}(n,m_1,m_2,m_3) \\
H^{(2,1)}(n,m_1,m_2,m_3) &= -\frac{\partial H^{(0,1)}(n,m_1,m_2,m_3)}{\partial m_2^2} + 2\frac{m_3-m_1-m_2}{m_2} H^{(0,2)}(n,m_1,m_2,m_3) \\
H^{(3,1)}(n,m_1,m_2,m_3) &= -\frac{\partial H^{(0,1)}(n,m_1,m_2,m_3)}{\partial m_3^2} - 2\frac{m_3-m_1-m_2}{m_3} H^{(0,2)}(n,m_1,m_2,m_3) \ .
\end{aligned}
\tag{34}
$$

As the pole terms in $(n-4)$ expansion of $H^{(i,1)}(n,m_1,m_2,m_3)$, $i=1,2,3$, are known to vanish [1], from Eq.(34) and with the result in Eq.(31) we deduce that $H^{(0,2)}(n,m_1,m_2,m_3)$ has no pole at $n=4$; the Eq.(34) takes the form for $n=4$

$$
\begin{aligned}
H^{(1,1)}(n=4,m_1,m_2,m_3) &= C^2(n)\Bigg[-\frac{\partial H^{(0,1)}(n=4,m_1,m_2,m_3)}{\partial m_1^2} \\
&\quad +2\frac{m_3-m_1-m_2}{m_1} H^{(0,2)}(n=4,m_1,m_2,m_3)\Bigg] \\
H^{(2,1)}(n=4,m_1,m_2,m_3) &= C^2(n)\Bigg[-\frac{\partial H^{(0,1)}(n=4,m_1,m_2,m_3)}{\partial m_2^2} \\
&\quad +2\frac{m_3-m_1-m_2}{m_2} H^{(0,2)}(n=4,m_1,m_2,m_3)\Bigg] \\
H^{(3,1)}(n=4,m_1,m_2,m_3) &= C^2(n)\Bigg[-\frac{\partial H^{(0,1)}(n=4,m_1,m_2,m_3)}{\partial m_3^2} \\
&\quad -2\frac{m_3-m_1-m_2}{m_3} H^{(0,2)}(n=4,m_1,m_2,m_3)\Bigg] \ .
\end{aligned}
\tag{35}
$$

Therefore to find $H^{(i,1)}(n=4,m_1,m_2,m_3)$, $i=1,2,3$, it is enough to insert the derivatives in respect to the masses, easily obtained from Eq.(32), and to have the explicit form of $H^{(0,2)}(n=4,m_1,m_2,m_3)$. We evaluate it in the next section using dispersion relation method.

There is no further obstacle in obtaining higher order terms in the p^2 expansion at the pseudothreshold, as can be verified with an explicit algebraic calculation, by solving iteratively the system of linear equations, obtaining the coefficients of any order in terms of the coefficients of lower orders. The reason is that the troublesome denominator $1/(n-4)$ appears only when expressing the first order coefficients in terms of the zeroth order coefficients; the denominators appearing at higher orders are $1/(n-5), 1/(n-6), \cdots$, regular at $n=4$.

The explicit expressions of the solutions are easily obtained with an algebraic program, but they are somewhat lengthy and we do not present them here for the sake of brevity. Let us also observe that for the practical purposes of solving the system of differential equations numerically only the values at pseudothresholds and first derivatives are needed.

It is worth to note that the functions $G_i^0(m_1,m_2,m_3)$, $i=1,2,3$ are not independent (the pole terms in Eq.(33) have to cancel) and the relation between them, using Eq.(15) and Eq.(29), reads

$$
\begin{aligned}
0 = \quad & G_1^{(0)}(m_1,m_2,m_3)(-8m_1-4m_2+4m_3) \\
+ \quad & G_2^{(0)}(m_1,m_2,m_3)(-4m_1-8m_2+4m_3) \\
+ \quad & G_3^{(0)}(m_1,m_2,m_3)(-4m_1-4m_2+8m_3) \\
+ \quad & \frac{1}{2}\Big[(-m_1-m_2+m_3) \\
& \qquad + \log(m_1^2)(-m_2+m_3) + \log(m_2^2)(-m_1+m_3) - \log(m_3^2)(m_1+m_2)\Big] \\
+ \quad & \frac{1}{8}\Big[\log^2(m_1^2)(2m_1+m_2-m_3) \\
& \qquad + \log^2(m_2^2)(m_1+2m_2-m_3) + \log^2(m_3^2)(m_1+m_2-2m_3)\Big] \\
+ \quad & \frac{1}{4}\Big[\log(m_1^2)\log(m_2^2)(m_1+m_2) \\
& \qquad + \log(m_1^2)\log(m_3^2)(m_1-m_3) + \log(m_2^2)\log(m_3^2)(m_2-m_3)\Big] \ .
\end{aligned} \tag{36}
$$

This relation is of course fulfilled by the already known solutions Eq.(29).

5 The value of $H^{(0,2)}(n=4,m_1,m_2,m_3)$.

Let us define the function $\tilde{F}(p^2)$ through the equation

$$
\begin{aligned}
F_0(n=4,m_1^2,m_2^2,m_3^2,p^2) = & \qquad F_0(n=4,m_1^2,m_2^2,m_3^2,0) \\
& + \ p^2 \ F_0'(n=4,m_1^2,m_2^2,m_3^2,0) \\
& + \qquad \tilde{F}(p^2) \ .
\end{aligned} \tag{37}
$$

The subtracted dispersion relation for $\tilde{F}(p^2)$ reads

$$\tilde{F}(p^2) = (p^2)^2 \int\limits_{(m_1+m_2+m_3)^2}^{\infty} \frac{du}{u^2(u+p^2)} \; \frac{1}{\pi} \; \mathrm{Im}F_0(n=4, m_1^2, m_2^2, m_3^2, u) \ , \tag{38}$$

where

$$\frac{1}{\pi} \; \mathrm{Im}F_0(n=4, m_1^2, m_2^2, m_3^2, u) = \frac{1}{16} \int\limits_{(m_1+m_2)^2}^{(\sqrt{u}-m_3)^2} db \; \frac{R(u,b,m_3^2)}{u} \; \frac{R(b,m_1^2,m_2^2)}{b} \ , \tag{39}$$

with the usual definition of the function R

$$R(x,y,z) = \sqrt{x^2+y^2+z^2-2xy-2xz-2yz} \ . \tag{40}$$

The function $H^{(0,2)}(n=4, m_1, m_2, m_3)$ which we want to calculate in this section in the notation of Eq.(37),Eq.(38) reads

$$\begin{aligned} H^{(0,2)}(n=4, m_1, m_2, m_3) = \; & \frac{1}{2}\tilde{F}''(p^2 = -(m_3-m_1-m_2)^2) \quad = \\ \int\limits_{(m_1+m_2+m_3)^2}^{\infty} & \frac{du}{(u-(m_3-m_1-m_2)^2)^3} \; \frac{1}{\pi} \; \mathrm{Im}F_0(n=4, m_1^2, m_2^2, m_3^2, u) \ , \end{aligned} \tag{41}$$

By using Eq.(39) and exchanging the order of integration we get

$$\begin{aligned} & H^{(0,2)}(n=4, m_1, m_2, m_3) = \\ & \frac{1}{16} \int\limits_{(m_1+m_2)^2}^{\infty} db \; \frac{R(b,m_1^2,m_2^2)}{b} \int\limits_{(\sqrt{b}+m_3)^2}^{\infty} du \frac{R(u,b,m_3^2)}{u \; (u-(m_3-m_1-m_2)^2)^3} \quad . \end{aligned} \tag{42}$$

The integration in u is elementary and gives

$$H^{(0,2)}(n=4, m_1, m_2, m_3) = \frac{1}{32} \int\limits_{(m_1+m_2)^2}^{\infty} db \; \frac{R(b,m_1^2,m_2^2)}{b} \tilde{H}(b) \ , \tag{43}$$

where

$$
\begin{aligned}
\tilde{H}(b) \;=\; & \frac{\log(y_S)}{\mathcal{R}(b)(m_3-m_1-m_2)^2}\left[-\frac{2(m_3^2-b)^2}{(m_3-m_1-m_2)^4}+\frac{2(b+m_3^2)}{(m_3-m_1-m_2)^2}+\frac{4bm_3^2}{\mathcal{R}^2(b)}\right] \\
& +\log\left(\frac{b}{m_3^2}\right)\frac{-b+m_3^2}{(m_3-m_1-m_2)^6} \\
& +\frac{2}{(m_3-m_1-m_2)^4}+\frac{1}{(m_3-m_1-m_2)^2}\frac{b+m_3^2}{\mathcal{R}^2(b)}-\frac{1}{\mathcal{R}^2(b)}\ , \qquad (44)
\end{aligned}
$$

with

$$
\mathcal{R}(b) = R\left(b,(m_1+m_2-m_3)^2,m_3^2\right) \qquad (45)
$$

and

$$
y_S = \frac{b+m_3^2-(m_1+m_2-m_3)^2-\mathcal{R}(b)}{2\sqrt{b}\,m_3}\ . \qquad (46)
$$

To perform the last integration, it is convenient to differentiate with respect to one of the masses Eq.(43) in its integral form, to perform the integration in b of the derivative so obtained (the integration in b being now elementary) and then to integrate again in the mass. As a final result we find

$$
\begin{aligned}
& H^{(0,2)}(n=4,m_1,m_2,m_3) = \frac{1}{32}\left\{\frac{R^2(m_1,m_2,-m_3)}{2\,(m_3-m_1-m_2)^4}\right. \\
& +\frac{2\,m_1\ \log\left(\frac{m_1}{m_3}\right)}{(m_3-m_1-m_2)^5}\left(m_2^2+m_3^2-m_1m_2+m_1m_3\right) \\
& +\frac{2\,m_2\ \log\left(\frac{m_2}{m_3}\right)}{(m_3-m_1-m_2)^5}\left(m_1^2+m_3^2-m_1m_2+m_2m_3\right) \\
& +\frac{2\,\mathcal{I}_0}{(m_3-m_1-m_2)^6}\left(2m_1^2m_2^2-m_1^2m_3^2-m_2^2m_3^2\right)-\frac{2\,m_3^2\,\mathcal{I}_1}{(m_3-m_1-m_2)^6}\left(m_1^2-m_2^2\right) \\
& \left.-\frac{4\,\mathcal{I}_2}{(m_3-m_1-m_2)^3}\ \frac{m_1m_2m_3}{R\left((m_1+m_2)^2,(m_1-m_3)^2,(m_2-m_3)^2\right)}\right\}\ , \qquad (47)
\end{aligned}
$$

where the integrals $\mathcal{I}_i \;\; i=0,1,2$ are defined in the following way

$$
\mathcal{I}_0 \;=\; \int_{(m_1+m_2)^2}^{\infty} db\ \left[\frac{\log(y_S)}{R\left(b,(m_1-m_3)^2,(m_2-m_3)^2\right)}+\frac{\log\left(\frac{\sqrt{b}}{m_3}\right)}{R(b,m_1^2,m_2^2)}\right]
$$

$$= \quad \zeta_2 + \log\left(\frac{m_3}{m_1}\right)\log\left(\frac{m_2}{m_3}\right) - \mathrm{Li}_2\left(1-\frac{m_1}{m_3}\right) - \mathrm{Li}_2\left(1-\frac{m_2}{m_3}\right) \ , \tag{48}$$

$$\mathcal{I}_1 = \int\limits_{(m_1+m_2)^2}^{\infty} \frac{db}{b} \left[\frac{(m_1-m_2)(m_1+m_2-2m_3)}{R\left(b,(m_1-m_3)^2,(m_2-m_3)^2\right)}\log(y_S) - \frac{(m_1^2-m_2^2)}{R(b,m_1^2,m_2^2)}\log\left(\frac{\sqrt{b}}{m_3}\right) \ \right]$$
$$= -\zeta_2 + \log\left(\frac{m_1}{m_2}\right)\log\left(\frac{m_2 m_3}{(m_1+m_2)^2}\right) - 2\mathrm{Li}_2\left(-\frac{m_1}{m_2}\right) + \mathrm{Li}_2\left(1-\frac{m_1}{m_3}\right) - \mathrm{Li}_2\left(1-\frac{m_2}{m_3}\right) \ , \tag{49}$$

and

$$\mathcal{I}_2 \ = \ \int\limits_{(m_1+m_2)^2}^{\infty} \frac{db}{b-(m_1+m_2)^2} \ \frac{R\left((m_1+m_2)^2,(m_1-m_3)^2,(m_2-m_3)^2\right)}{R\left(b,(m_1-m_3)^2,(m_2-m_3)^2\right)} \ \log(y_S)$$
$$\begin{aligned} = & -5\zeta_2 + \log\left(\frac{t_2^2\ (1+t_1^2)}{1+t_2^2}\right)\ \log\left(\frac{1-t_1^2\ t_2^2}{1+t_2^2}\right) - \log\left(\frac{1-t_1\ t_2}{1-t_2}\right)\ \log\left(\frac{t_2^2\ (1-t_1)^2}{(1-t_2)^2}\right) \\ & - \ 2\ \mathrm{Li}_2\left(\frac{t_2\,(1-t_1)}{t_2-1}\right) \ - 2\ \mathrm{Li}_2\left(\frac{t_1-t_2}{1-t_2}\right) + \ 2\ \mathrm{Li}_2\left(\frac{1+t_1\ t_2}{1+t_2}\right) \ - 2\ \mathrm{Li}_2\left(\frac{t_1+t_2}{1+t_2}\right) \\ & + \ \mathrm{Li}_2\left(\frac{t_2^2\ (1+t_1^2)}{1+t_2^2}\right) + \ \mathrm{Li}_2\left(\frac{1-t_2^2}{1+t_2^2}\right) + \ \mathrm{Li}_2\left(\frac{t_2^2-t_1^2}{1+t_2^2}\right) - \ \mathrm{Li}_2\left(\frac{t_2^2-1}{1+t_2^2}\right) \ , \end{aligned} \tag{50}$$

where

$$t_1 = \frac{\sqrt{m_1+m_2-m_3}-\sqrt{m_3}}{\sqrt{m_1+m_2-m_3}+\sqrt{m_3}} \quad \text{and} \quad t_2 = \frac{\sqrt{m_1}-\sqrt{m_2}}{\sqrt{m_1}+\sqrt{m_2}} \ . \tag{51}$$

The last expression was found assuming $m_1 > m_2 > m_3$, but the analytic continuation to other regions is straightforward.

6 Summary

In this paper we have presented the expansion of the 2-loop sunrise selfmass master amplitudes at the pseudothreshold $p^2 = -(m_1+m_2-m_3)^2$. The other pseudothresholds can be easily found by the permutation of the masses. We define the expansion in Eq.(8); the values of the amplitudes at the pseudothreshold are given in Eq.(15), Eq.(16), Eq.(25) and

Eq.(29). The first order terms in the pseudothreshold expansion of the master amplitudes at ($n = 4$) are presented in Eq.(31), Eq.(32) and Eq.(35), while the second order term of $F_0(n = 4, m_1^2, m_2^2, m_3^2, p^2 = -(m_1 + m_2 - m_3)^2)$ is presented in Eq.(47). The higher order terms, which are not given explicitly here, can be easily found by solving recursively, at each order, a system of four algebraic linear equations. The expansion at the pseudothreshold cannot be simply deduced from the known expansion at the threshold [7] and vice versa, even if at first sight they seem to be connected by the change of sign $m_3 \to -m_3$. In fact the analytic properties of the amplitudes are different at the two points: at the pseudothreshold the sunrise amplitudes are regular, so that the solution of the Eq.(4) can be expanded as a single power series, while at the threshold the sunrise amplitudes develop a branch point and its expansion is indeed the sum of two series, [8].

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7 Appendix.

In this appendix are given the definitions of the functions used in the text.

Function $g(m_1, m_2, m_3)$ used in Eq.(17)

$$\begin{aligned} g(m_1, m_2, m_3) &= r_0 + r_1 \log(m_1^2) + r_2 \log(m_2^2) + r_3 \log(m_3^2) + r_{1,1} \log^2(m_1^2) \\ &\quad + r_{1,2} \log(m_1^2) \log(m_2^2) + r_{1,3} \log(m_1^2) \log(m_3^2) + r_{2,2} \log^2(m_2^2) \\ &\quad + r_{2,3} \log(m_2^2) \log(m_3^2) + r_{3,3} \log^2(m_3^2) \ , \end{aligned} \tag{52}$$

where

$$\begin{aligned} r_0 = &-\frac{11}{16}x^5 - \frac{147}{32}(m_1 + m_2)x^4 - (\frac{485}{64}m_1^2 + \frac{211}{16}m_1 m_2 + \frac{485}{64}m_2^2)x^3 \\ &-(\frac{81}{16}m_1^3 + \frac{839}{64}m_1^2 m_2 + \frac{839}{64}m_1 m_2^2 + \frac{81}{16}m_2^3)x^2 \\ &-(\frac{21}{16}m_1^4 + 5m_1^3 m_2 + \frac{13}{2}m_1^2 m_2^2 + 5m_1 m_2^3 + \frac{21}{16}m_2^4)x \\ &-\frac{7}{16}m_1 m_2(m_1^3 + 2m_1^2 m_2 + 2m_1 m_2^2 + m_2^3) \end{aligned}$$

$$
\begin{aligned}
r_1 &= m_1^2\Big[\frac{1}{2}x^3+\frac{9}{8}(m_1+m_2)x^2+(\frac{9}{16}m_1^2+\frac{5}{4}m_1m_2+\frac{5}{16}m_2^2)x+\frac{3}{16}m_1m_2(m_1+m_2)\Big] \\
r_2 &= m_2^2\Big[\frac{1}{2}x^3+\frac{9}{8}(m_1+m_2)x^2+(\frac{5}{16}m_1^2+\frac{5}{4}m_1m_2+\frac{9}{16}m_2^2)x+\frac{3}{16}m_1m_2(m_1+m_2)\Big] \\
r_3 &= \frac{15}{4}x^5+\frac{71}{8}(m_1+m_2)x^4+(\frac{121}{16}m_1^2+\frac{67}{4}m_1m_2+\frac{121}{16}m_2^2)x^3 \\
&\quad +(3m_1^3+\frac{171}{16}m_1^2m_2+\frac{171}{16}m_1m_2^2+3m_2^3)x^2 \\
&\quad +(\frac{9}{16}m_1^4+\frac{11}{4}m_1^3m_2+\frac{35}{8}m_1^2m_2^2+\frac{11}{4}m_1m_2^3+\frac{9}{16}m_2^4)x+\frac{3}{16}m_1m_2(m_1+m_2)^3 \\
r_{1,1} &= -\frac{3}{16}m_1^2\left(m_3-\frac{1}{2}(m_1+m_2)\right)\left(m_3^2-m_3(m_1+m_2)+\frac{1}{3}m_1m_2\right) \\
r_{1,2} &= \frac{1}{8}m_1^2m_2^2\left(m_3-\frac{1}{2}(m_1+m_2)\right) \\
r_{1,3} &= -\frac{1}{8}m_1^2\left(m_3-\frac{1}{2}(m_1+m_2)\right)\left(3x^2+3(m_1+m_2)x+m_2(m_1+m_2)\right) \\
r_{2,2} &= -\frac{3}{16}m_2^2\left(m_3-\frac{1}{2}(m_1+m_2)\right)\left(m_3^2-m_3(m_1+m_2)+\frac{1}{3}m_1m_2\right) \\
r_{2,3} &= -\frac{1}{8}m_2^2\left(m_3-\frac{1}{2}(m_1+m_2)\right)\left(3x^2+3(m_1+m_2)x+m_1(m_1+m_2)\right) \\
r_{3,3} &= -\frac{1}{16}\left(m_3-\frac{1}{2}(m_1+m_2)\right) \\
&\quad \left(3(m_1^2+m_2^2)x^2+3(m_1^3+m_1^2m_2+m_1m_2^2+m_2^3)x+m_1m_2(m_1+m_2)^2\right) \qquad (53)
\end{aligned}
$$

and $x = m_3 - m_1 - m_2$.

The definition of the coefficients used in Eq.(24):

$$
\begin{aligned}
q_0 &= m_1^2m_2^2(m_1+m_2)\Big[\frac{7}{48}(m_1^2+m_1m_2+m_2^2)-\frac{1}{16}m_1^2\log(m_1^2)-\frac{1}{16}m_2^2\log(m_2^2) \\
&\quad +\frac{1}{96}(m_1\log(m_1^2)-m_2\log(m_2^2))^2\Big] \\
q_1 &= m_1^2m_2^2(m_1+m_2)^2\Big[-\frac{1}{8}(m_1+m_2)+\frac{1}{24}m_1\log(m_1^2)+\frac{1}{24}m_2\log(m_2^2)\Big] \\
q_2 &= \frac{1}{24}m_1^2m_2^2(m_1+m_2)^3 \\
q_{-1,0} &= m_1m_2\Big[-\frac{1}{192}(5m_1^4+29m_1^3m_2+104m_1^2m_2^2+29m_1m_2^3+5m_2^4) \\
&\quad +\frac{1}{48}m_1^2(m_1^2+5m_2^2)\log(m_1^2)+\frac{1}{48}m_2^2(5m_1^2+m_2^2)\log(m_2^2) \\
&\quad +\frac{1}{48}m_1m_2(m_1\log(m_1^2)-m_2\log(m_2^2))^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
q_{-1,1} &= m_1^2 m_2^2 (m_1+m_2)\Big[-\frac{7}{24}(m_1+m_2)+\frac{1}{12}\left(m_1\log(m_1^2)+m_2\log(m_2^2)\right)\Big] \\
q_{-1,2} &= \frac{1}{12}m_1^2 m_2^2 (m_1+m_2)^2 \\
q_{0,0} &= -\frac{5}{64}m_1^5-\frac{21}{32}m_1^4 m_2-\frac{67}{48}m_1^3 m_2^2-\frac{67}{48}m_1^2 m_2^3-\frac{21}{32}m_1 m_2^4-\frac{5}{64}m_2^5 \\
&\quad +m_1^2(\frac{1}{16}m_1^3+\frac{5}{24}m_1^2 m_2+\frac{11}{24}m_1 m_2^2+\frac{5}{16}m_2^3)\log(m_1^2) \\
&\quad +m_2^2(\frac{5}{16}m_1^3+\frac{11}{24}m_1^2 m_2+\frac{5}{24}m_1 m_2^2+\frac{1}{16}m_2^3)\log(m_2^2) \\
&\quad -\frac{1}{16}m_1^2 m_2^2(m_1+m_2)\log(m_1^2)\log(m_2^2) \\
q_{0,1} &= m_1 m_2\Big[-\frac{1}{6}m_1^3+\frac{13}{24}m_1^2 m_2+\frac{13}{24}m_1 m_2^2-\frac{1}{6}m_2^3 \\
&\quad +\frac{1}{8}m_1 m_2(m_1+m_2)(\log(m_1^2)+\log(m_2^2))\Big] \\
q_{0,2} &= \frac{1}{4}m_1^2 m_2^2(m_1+m_2) \\
q_{1,0} &= \frac{65}{64}m_1^4+\frac{1031}{192}m_1^3 m_2+\frac{857}{96}m_1^2 m_2^2+\frac{1031}{192}m_1 m_2^3+\frac{65}{64}m_2^4 \\
&\quad - m_1^2(\frac{3}{16}m_1^2+\frac{5}{12}m_1 m_2+\frac{7}{16}m_2^2)\log(m_1^2)-m_2^2(\frac{7}{16}m_1^2+\frac{5}{12}m_1 m_2+\frac{3}{16}m_2^2)\log(m_2^2) \\
&\quad -\frac{3}{32}(m_1+m_2)^2\Big[m_1^2\log^2(m_1^2)+m_2^2\log^2(m_2^2)\Big]+\frac{1}{8}m_1^2 m_2^2\log(m_1^2)\log(m_2^2) \\
q_{1,1} &= m_1^4+\frac{137}{24}m_1^3 m_2+\frac{25}{3}m_1^2 m_2^2+\frac{137}{24}m_1 m_2^3+m_2^4 \\
&\quad -m_1^2(\frac{3}{8}m_1^2+\frac{3}{4}m_1 m_2+\frac{5}{8}m_2^2)\log(m_1^2)-m_2^2(\frac{5}{8}m_1^2+\frac{3}{4}m_1 m_2+\frac{3}{8}m_2^2)\log(m_2^2) \\
q_{1,2} &= -\frac{1}{8}\left(3m_1^4+6m_1^3 m_2+10m_1^2 m_2^2+6m_1 m_2^3+3m_2^4\right) \\
q_{2,0} &= -(m_1+m_2)\Big[\frac{193}{64}m_1^2+\frac{1775}{192}m_1 m_2+\frac{193}{64}m_2^2 \\
&\quad +\frac{3}{8}m_1^2\log(m_1^2)+\frac{3}{8}m_2^2\log(m_2^2)-\frac{9}{32}m_1^2\log^2(m_1^2)-\frac{9}{32}m_2^2\log^2(m_2^2)\Big] \\
q_{2,1} &= (m_1+m_2)\Big[-\frac{63}{8}m_1^2-\frac{433}{24}m_1 m_2-\frac{63}{8}m_2^2+\frac{9}{8}m_1^2\log(m_1^2)+\frac{9}{8}m_2^2\log(m_2^2)\Big] \\
q_{2,2} &= \frac{9}{8}(m_1+m_2)(m_1^2+m_2^2) \\
q_{3,0} &= \frac{251}{64}m_1^2+\frac{241}{24}m_1 m_2+\frac{251}{64}m_2^2 \\
&\quad +\frac{1}{2}m_1^2\log(m_1^2)+\frac{1}{2}m_2^2\log(m_2^2)-\frac{3}{16}m_1^2\log^2(m_1^2)-\frac{3}{16}m_2^2\log^2(m_2^2)
\end{aligned}
$$

$$
\begin{aligned}
q_{3,1} &= \frac{153}{8}m_1^2 + 39m_1m_2 + \frac{153}{8}m_2^2 - \frac{3}{4}m_1^2\log(m_1^2) - \frac{3}{4}m_2^2\log(m_2^2) \\
q_{3,2} &= -\frac{3}{4}(m_1^2 + m_2^2) \\
q_{4,0} &= -\frac{37}{32}(m_1 + m_2) \quad q_{4,1} = -\frac{79}{4}(m_1 + m_2) \quad q_{5,0} = -\frac{11}{16} \quad q_{5,1} = \frac{15}{2} \qquad (54)
\end{aligned}
$$

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